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LETTER TO THE EDITOR

The Bethe ansatz as a matrix product ansatz**Francisco C Alcaraz and Matheus J Lazo**Instituto de Física de São Carlos, Universidade de São Paulo, Caixa Postal 369,
13560-590 São Carlos, São Paulo, Brazil

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Online at stacks.iop.org/JPhysA/37/L1 (DOI: 10.1088/0305-4470/37/1/L01)**Abstract**

The Bethe ansatz in its several formulations is a common tool for the exact solution of one-dimensional quantum Hamiltonians. This ansatz asserts that several eigenfunctions of the Hamiltonians are given in terms of a sum of permutations of plane waves. We present results that induce us to expect that, alternatively, the eigenfunctions of all the exact integrable quantum chains can also be expressed by a matrix product ansatz. In this ansatz several components of the eigenfunctions are obtained through the algebraic properties of properly defined matrices. This ansatz allows an unified formulation of several exact integrable Hamiltonians. We show how to formulate this ansatz for a large family of quantum chains such as the anisotropic Heisenberg model, Fateev–Zamolodchikov model, Izergin–Korepin model, Sutherland model, t – J model, Hubbard model, etc.

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Since the pioneering work of Bethe in 1931 [1] the Bethe ansatz and its generalizations have proved to be quite an efficient tool in the description of the eigenvectors of a large variety of one-dimensional quantum chains and two-dimensional transfer matrices (see, e.g., [2–5] for reviews). On the other hand, in the last two decades [6–9] it has been shown that some special quantum chains, although not integrable through the Bethe ansatz, have the components of their ground-state wavefunctions given in terms of a product of matrices. In this matrix product ansatz (MPA), apart from a normalization constant, these components are fixed by the algebraic properties of the matrices defining the MPA. In another context an MPA has also been applied quite successfully to the evaluation of the stationary distribution of probabilities of some stochastic models in one dimension [10]. The time fluctuations of these stochastic models are described by the ground-state wavefunction of a related spin Hamiltonian. The simplest example is the one-dimensional asymmetric exclusion process [10] whose related spin Hamiltonian is the anisotropic Heisenberg chain, or XXZ chain, with appropriate boundary fields [11]. The stationary properties of the model are given in terms of the algebraic relations of the matrices appearing on the MPA. This ansatz was used in a variety of problems including

interface growth [12], boundary induced phase transitions [10, 13–15], the dynamics of shocks [16] or traffic flow [17].

An important development of the MPA that appeared in the context of stochastic models is the dynamical matrix product ansatz (DMPA) [18, 19]. This *ansatz* allows, whenever it is valid, the calculation of the probability densities of the stochastic system at arbitrary times. In the related spin Hamiltonian this DMPA asserts that not only the ground-state wavefunction, as in the standard MPA, but also an arbitrary wavefunction has components expressed in terms of a matrix product ansatz whose matrices, in distinction from the standard MPA, are now time dependent. This DMPA was shown originally [18, 20] to be valid for the problem of asymmetric diffusion of particles on the lattice. More recently [21] (see also [22]) it was also shown that this DMPA can also be formulated in the context of stochastic models with two species of hard-core particles. The condition of validity of the DMPA reproduces the subspace of parameters where the model is known to be exactly integrable. This fact induced us to conjecture that all Hamiltonians, whether or not related to stochastic models, which are solvable through the Bethe ansatz may also be solvable by an appropriate MPA. This would mean that the components of the eigenfunctions of the exact integrable models, which according to the Bethe ansatz are normally given by a combination of plane waves, can also be obtained from the algebraic properties of the matrices defining the MPA.

In this letter, we are going to show how the eigenspectra of the exact integrable quantum chains, with global conservation laws, can be obtained by an appropriate matrix-product ansatz. In this way we were able to obtain the integrability of several well known exact integrable models. Among the models with one global conservation law we have the XXZ chain [23], the spin- S Fateev–Zamolodchikov model [24], the Izergin–Korepin model [25], the solvable spin-1 model of [26], etc, and among the models with two global conservations we have the supersymmetric t – J model [27], the spin-1 Sutherland and Perk–Schultz models [28, 29], the Hubbard model [30], as well the two-parameter integrable model presented in [33]. Moreover the matrix-product ansatz we propose enable us to show, with little effort, how to extend the above-mentioned models by including an arbitrary range of hard-core interactions without losing their exact integrability.

For brevity and in order to illustrate the proposed MPA we are going to present here two examples: the solution of the XXZ chain with arbitrary hard-core interactions among the up spins as an example of a model with one global conservation law, and the solution of the Hubbard model as an example of a model with two global conservations laws.

The Hamiltonian of the XXZ chain with a hard-core exclusion of S sites ($S = 1, 2, \dots$) is the anisotropic Heisenberg chain (anisotropy Δ) where any two up spins are not allowed to occupy lattice sites at distances smaller than S . This Hamiltonian in a L -sites periodic chain is given by

$$H_S = -\mathcal{P}_S \sum_{i=1}^L \frac{1}{2} (\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \Delta \sigma_i^z \sigma_{i+S}^z) \mathcal{P}_S + \frac{L\Delta}{2} \quad (1)$$

where $\sigma^x, \sigma^y, \sigma^z$ are spin- $\frac{1}{2}$ Pauli matrices and \mathcal{P}_S is a projector that projects out the configurations where any two up spins are at distances smaller than S . The case $S = 1$ corresponds to the standard exactly solvable XXZ chain [23]. The conserved charge associated with the global conservation law of (1) is the z -component of the total magnetization, or equivalently the number of up spins (particles). The translation invariance of the lattice also ensures that the momentum is also a good quantum number.

The ansatz we propose states that any of the wavefunctions $|\psi_{n,P}\rangle$ in the sector with n spins up ($n = 0, 1, 2, \dots$) and momentum $P = \frac{2\pi l}{L}$ ($l = 0, 1, 2, \dots, L - 1$) is given by a

matrix product ansatz, i.e., their amplitudes are given by the trace of the matrix product:

$$|\psi_{n,P}\rangle = \sum_{x_1, \dots, x_n}^* \text{Tr}(E^{x_1-1} A E^{x_2-x_1-1} A \dots E^{x_n-x_{n-1}-1} A E^{L-x_n} \Omega_P) |x_1, \dots, x_n\rangle \quad (2)$$

where $|x_1, \dots, x_n\rangle$ denotes the configurations with up spins at (x_1, \dots, x_n) and the symbol $(*)$ in the sum means the restriction to the configurations where $|x_{i+1} - x_i| \geq S$. The algebraic properties of the matrices A , E and Ω_P will be fixed by the eigenvalue equation

$$H_S |\psi_{n,P}\rangle = \varepsilon_n |\psi_{n,P}\rangle. \quad (3)$$

The algebraic relations of the matrices A and E with Ω_P will fix the momentum of the eigenfunction. The fact that $|\psi_{n,P}\rangle$ has a momentum P implies that the ratio of the amplitudes corresponding to configurations $|x_1, \dots, x_n\rangle$ and $|x_1+1, \dots, x_n+1\rangle$ is e^{-iP} and consequently from 2 we have the following relations:

$$A \Omega_P = e^{-iP} \Omega_P A \quad E \Omega_P = e^{-iP} \Omega_P E. \quad (4)$$

The eigenvalue equation (3) when applied to the components of $|\psi_{n,P}\rangle$ where all the up spins are at distances larger than S gives us the constraint

$$\begin{aligned} \varepsilon_n \text{Tr}(E^{x_1-1} A E^{x_2-x_1-1} \dots A E^{x_j-x_{j-1}-1} A E^{x_{j+1}-x_j-1} A \dots A E^{L-x_n} \Omega_P) \\ = \sum_{i=1}^n [\text{Tr}(E^{x_1-1} A E^{x_2-x_1-1} \dots A E^{x_i-x_{i-1}-2} A E^{x_{i+1}-x_i} A \dots A E^{L-x_n} \Omega_P) \\ + \text{Tr}(E^{x_1-1} A E^{x_2-x_1-1} \dots A E^{x_i-x_{i-1}} A E^{x_{i+1}-x_i-2} A \dots A E^{L-x_n} \Omega_P)] \end{aligned} \quad (5)$$

A convenient solution of this last equation is obtained by identifying the matrices A as composed by n spectral-parameter-dependent matrices

$$A = \sum_{j=1}^n A_{k_j} E^{2-S} \quad (6)$$

where the matrices A_{k_j} obey the commutation relations

$$E A_{k_j} = e^{ik_j} A_{k_j} E \quad A_{k_j} \Omega_P = e^{-iP(S-1)} \Omega_P A_{k_j} \quad (7)$$

and k_j ($j = 1, \dots, n$) are in general complex numbers unknown *a priori*. The energy and momentum are given, respectively, by

$$\varepsilon_n = - \sum_{j=1}^n (e^{ik_j} + e^{-ik_j} - 2\Delta) \quad P = \sum_{i=1}^n k_j. \quad (8)$$

The eigenvalue equation when applied to the other components of $|\psi_{n,P}\rangle$ will give the commutation relations of the matrices $\{A_{k_j}\}$ among themselves. In fact this algebra is obtained from the components where any pair of up spins are located at the closest positions ('matching conditions') x_j and $x_{j+1} = x_j + S$, namely

$$\begin{aligned} A_{k_j} A_{k_l} = S(k_j, k_l) A_{k_l} A_{k_j} \quad (j \neq l) \quad A_{k_j}^2 = 0 \\ S(k_j, k_l) = - \frac{e^{i(k_j+k_l)} + 1 - 2\Delta e^{ik_j}}{e^{i(k_j+k_l)} + 1 - 2\Delta e^{ik_l}} \quad j, l = 1, \dots, n. \end{aligned} \quad (9)$$

These last relations coincide with the Zamolodchikov algebra of creation operators in a (1+1)-dimensional field theory [31, 32]. No new algebraic relations appear for the matrices A_{k_j} and the associativity of the algebra (7) and (9) follows from the property $S(k_i, k_j) S(k_j, k_i) = 1$.

The cyclic property of the trace in (2), together with the algebraic relations (9), gives us, for each k_j ($j = 1, \dots, n$), the conditions

$$\begin{aligned} \text{Tr}(A_{k_1} \cdots A_{k_n} E^{L-n(S-1)} \Omega_P) \\ = e^{-ik_j(L-nS+n)} e^{-iP(S-1)} \left(\prod_{l=1}^n S(k_j, k_l) \right) \text{Tr}(A_{k_1} \cdots A_{k_n} E^{L-n(S-1)} \Omega_P) \end{aligned}$$

that fix the up-to-now free complex spectral parameters $\{k_j\}$

$$e^{ik_j L} = \prod_{l=1}^n S(k_j, k_l) e^{i(k_j - k_l)(S-1)} \quad (j = 1, \dots, n). \quad (10)$$

The wavefunctions, obtained by using (2), (8) and (9), can be written as a combination of plane waves ('wave numbers' $\{k_j\}$), and apart from a normalization constant coincide with those obtained by the standard Bethe ansatz [23] for $S = 1$ or arbitrary values of S [34].

As a second example we consider the standard Hubbard model whose Hamiltonian in a periodic lattice with L sites is given by

$$H_h = -t \sum_{j,\sigma} (c_{j,\sigma}^+ c_{j+1,\sigma} + c_{j+1,\sigma}^+ c_{j,\sigma}) + u \sum_j \eta_{j,-} \eta_{j,+} \quad (11)$$

where $c_{j,\sigma}^+$ are creation operators of electrons of spin $\sigma = \pm$ at site j , and $\eta_{j,\sigma} = \sum_{ja} c_{j,\sigma}^+ c_{j,\sigma}$ are the number operators of electrons of spin σ at site j . In this case we have two global conservation laws n_{\pm} , corresponding to the number of electrons with spin $\sigma = \pm$, and the total number of electrons is $n = n_+ + n_-$. In order to form our MPA we associate, as before, the matrix E with the empty sites, the matrices X^+ , X^- with the single occupied sites with electrons with spin up and down, respectively, and the matrix $X^0 \equiv X^+ E^{-1} X^-$ with the sites with double occupancy. Our MPA asserts that the components of the eigenfunction $|\psi_{n,P}\rangle$ of energy ϵ_n and momentum $P = 2\pi l/L$ ($l = 0, 1, \dots, L-1$) corresponding to the configuration $|x_1, Q_1, \dots, x_n, Q_n\rangle$ where the non-empty sites (x_1, \dots, x_n) have occupation (Q_1, \dots, Q_n) ($Q_i = +, -, 0$) will be given by the trace

$$\text{Tr}(E^{x_1-1} X^{Q_1} E^{x_2-x_1-1} X^{Q_2} \dots X^{Q_n} E^{L-x_n} \Omega_P). \quad (12)$$

The momentum of the state P , as in (4), is fixed by imposing the commutation relations $X^Q \Omega_P = e^{-iP} \Omega_P X^Q$ ($Q = +, -, 0$) and $E \Omega_P = e^{-iP} \Omega_P E$. The eigenvalue equation $H_h |\Psi_{n,P}\rangle = \epsilon_n |\Psi_{n,P}\rangle$ will provide the algebraic relations of the matrices E and X^Q .

The components of the wavefunction corresponding to the configurations where all the particles are at distances $|x_{i+1} - x_i| > 1$ will give a generalization of (2) whose solution is obtained by introducing the convenient n spectral-parameter-dependent matrices

$$X^Q = \sum_{j=1}^n E X_{k_j}^Q \quad (Q = +, -) \quad (13)$$

whose commutation relations with the matrices E and Ω_P are

$$E X_{k_j}^Q = e^{ik_j} X_{k_j}^Q E \quad X_{k_j}^Q \Omega_P = \Omega_P X_{k_j}^Q. \quad (14)$$

The energy and momentum in terms of these unknown complex spectral parameters are given, as in (8), by

$$\epsilon_n = - \sum_{j=1}^n (e^{ik_j} + e^{-ik_j}) \quad P = \sum_{j=1}^n k_j. \quad (15)$$

The components where the particles occupy the closest positions and those where we have double occupancy give us by using (14) and (15) the algebraic relations

$$\begin{aligned}
X_{k_l}^Q X_{k_j}^Q &= S_{QQ}^{QQ}(k_l, k_j) X_{k_j}^Q X_{k_l}^Q & (X_{k_j}^Q)^2 &= 0 \\
X_{k_l}^Q X_{k_j}^{Q'} &= S_{QQ'}^{QQ'}(k_l, k_j) X_{k_j}^{Q'} X_{k_l}^Q + S_{Q'Q}^{QQ'}(k_l, k_j) X_{k_j}^Q X_{k_l}^{Q'} \\
S_{QQ}^{QQ} &= -1 & S_{Q'Q}^{QQ'}(k_l, k_j) &= -u e^{i(k_l+k_j)} / \alpha \\
S_{Q'Q}^{QQ'}(k_l, k_j) &= t(e^{ik_l} - e^{ik_j})(1 + e^{i(k_l+k_j)}) / \alpha \\
\alpha &= u e^{i(k_l+k_j)} + t(1 + e^{i(k_l+k_j)})(e^{ik_l} - e^{ik_j})
\end{aligned} \tag{16}$$

where $Q' = -Q$ and $Q = \pm$. Relations (14) and (16) define completely the algebra whose structural constants are the well-known S -matrix of the Hubbard model [30]. Since the several components of the wavefunction should be uniquely related, the above algebra should be associative. This associativity implies that the above S -matrix should satisfy the Yang–Baxter relations [35, 2], which is indeed the case [30]. The components of the wavefunction corresponding to the configurations where we have three or four particles in next-neighbouring sites would give in principle new relations involving three or four operators $X_{k_j}^Q$. These new relations are however consequences of the above relations (14) and (16). The cyclic property of the trace in (12) and the algebraic relations (14) and (16) will imply

$$\begin{aligned}
&\text{Tr}(X_{k_1}^{Q_1} \dots X_{k_{j-1}}^{Q_{j-1}} X_{k_j}^{Q_j} \dots X_{k_n}^{Q_n} E^L \Omega_P) \\
&= (-1)^n e^{ik_j L} \sum_{Q'_1, \dots, Q'_n} \mathcal{T}(\{Q\}, \{Q'\}) \text{Tr}(X_{k_1}^{Q'_1} \dots X_{k_n}^{Q'_n} E^L \Omega_P)
\end{aligned} \tag{17}$$

where

$$\mathcal{T}(\{Q\}, \{Q'\}) = \sum_{Q''_1, \dots, Q''_n} \prod_{i=1}^n S_{Q_i, Q''_i}^{Q'_i, Q''_{i+1}}(k_i, k_j) \tag{18}$$

is the transfer matrix of a non-homogeneous six-vertex model defined on a cylinder of perimeter n and with Boltzmann weights given by the S -matrices defined in (16). The eigenvalues of this auxiliary problem $\Lambda_n(k_j; k_1, \dots, k_n)$ can be obtained in a standard way by the coordinate-Bethe ansatz [23] or by the quantum inverse scattering method [36]. Using these eigenvalues in relation (17) the spectral parameters $\{k_j\}$ will be fixed by the solutions of the system of equations

$$e^{-ik_j L} = (-1)^n \Lambda_n(k_j; k_1, \dots, k_n) \quad (j = 1, \dots, n). \tag{19}$$

These last equations coincide with the Bethe ansatz equations derived through the standard coordinate Bethe ansatz [30].

Generalizations of our MPA are quite simple to implement (a detailed version of this letter containing several generalizations will be presented elsewhere [37]). For example, the solution of the excluded volume Hubbard model where electrons with spin up (down) exclude other electrons at S_+ (S_-) sites on its right, but allow double occupancy at any site, is just obtained by changing in (13) $X^Q = \sum_j E^{S_{Q+1}} X_{k_j}^Q$ ($Q = \pm$).

We have also shown [37] that the above MPA also works for the other known exactly integrable models with two conserved global quantities ($U(1) \otimes U(1)$), like the Essler–Korepin–Schoutens model [38] or the generalized two-parameter integrable model introduced in [33]. In those last cases the same MPA presented above for the Hubbard model apply except that now the matrices X^0 associated with the sites with double occupancy are given in terms of

new operators $X^0 = Y^+ E^{-1} Y^-$, and we were able to rederive all the results obtained for these models through the standard Bethe ansatz. Hamiltonians with two global conservations that do not allow double occupancy like the stochastic Hamiltonian associated with the asymmetric diffusion with two kinds of particles [34], or the supersymmetric t - J quantum chain [27] or the $SU(3)$ Sutherland and Perk–Schultz models [28, 29], are obtained through the MPA (12), without the matrices X^0 .

Models of spin-1 with a single global conservation law (conservation of the z -magnetization) such as the Fateev–Zamolodchikov model [24], the Izergin–Korepin model [25], or the solvable spin-1 Hamiltonian introduced in [33], have their solutions given by a MPA similar [37] in (2), where now to that we associate the matrices E , A and $BE^{-1}B$ with the sites occupied by particles with spin $S^z = -1, 0$ or $+1$, respectively.

We have also obtained [37] an appropriate extension of the presented MPA to non-periodic, but exact integrable boundaries such as, for example, the XXZ chain with surface fields.

It is interesting to note that in the cases of exact integrable Hamiltonians associated with stochastic models, as in [11, 19], since we can write all eigenfunctions in a MPA, our results imply that we can equivalently write at any time the probability distribution of the model in terms of a time-dependent MPA, as happens in the DMPA [18].

In conclusion, we have shown that the eigenfunctions of a large variety of exact integrable quantum chains can be expressed in an unified way in terms of a matrix product ansatz, whose matrices satisfy an associative algebra. The associativity of the algebra that warrants the exact integrability of the model is a consequence of the Yang–Baxter relations. Our results indicate that all the exact solutions obtained through the coordinate Bethe ansatz can also be obtained through the present MPA. Conversely, all the new exact solutions obtained by this MPA probably can also be formulated by a suitable coordinate Bethe ansatz. The advantages of the present MPA in the search for new exact integrable models remain its simplicity and unifying implementation for arbitrary systems. For example, the MPA solution for integrable quantum chains with exclusion effects is quite simple. Even in the cases where more than one conservation law exists the MPA gives us an S -matrix independent of the excluded volume [37], in contrast to the standard Bethe ansatz where the S -matrix depends on the range of exclusion of particles [39].

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References

- [1] Bethe H A 1931 *Z. Phys.* **71** 205
- [2] Baxter R J 1982 *Exactly Solved Models in Statistical Mechanics* (New York: Academic)
- [3] Korepin V E, Izergin A G and Bogoliubov N M 1992 *Quantum Inverse Scattering Method, Correlation Functions and Algebraic Bethe Ansatz* (Cambridge: Cambridge University Press)
- [4] Essler F H L and Korepin V E 1994 *Exactly Solvable Models of Strongly Correlated Electrons* (Singapore: World Scientific)
- [5] Schlottmann P 1977 *Int. J. Mod. Phys. B* **11** 355
- [6] Affleck I, Kennedy T, Lieb E H and Tasaki H 1988 *Commun. Math. Phys.* **115** 477
- [7] Arovas D P, Auerbach A and Haldane F D M 1998 *Phys. Rev. Lett.* **60** 531
- [8] Fannes M, Nachtergaele B and Werner R F 1992 *Commun. Math. Phys.* **144** 443
- [9] Klumper A, Schadschneider A and Zittarz J 1992 *Z. Phys. B* **87** 281
- Klumper A, Schadschneider A and Zittarz J 1993 *Europhys. Lett.* **24** 293
- [10] Derrida B, Evans M R, Hakim V and Pasquier V 1993 *J. Phys. A: Math. Gen.* **26** 1493
- [11] Alcaraz F C, Droz M, Henkel M and Rittenberg V 1994 *Ann. Phys., NY* **230** 250

-
- [12] Krug J and Spohn H 1991 *Solids Far From Equilibrium* ed C Godrèche (Cambridge: Cambridge University Press)
- [13] Derrida B, Domany E and Mukamel D 1992 *J. Stat. Phys.* **69** 667
- [14] Domany R and Schütz E 1993 *J. Stat. Phys.* **72** 277
- [15] Evans M R, Foster D P, Godrèche C and Mukamel D 1995 *J. Stat. Phys.* **80** 69
- [16] Derrida B, Lebowitz J L and Speer E R 1997 *J. Stat. Phys.* **89** 135
- [17] Nagel K and Schreckenberg M 1992 *J. Physique* **2** 2221
- [18] Stinchcombe R B and Schütz G M 1995 *Phys. Rev. Lett.* **75** 140
Stinchcombe R B and Schütz G M 1995 *Europhys. Lett.* **29** 663
- [19] Schütz G M 2000 *Phase Transitions and Critical Phenomena* vol 19, ed C Domb and J Lebowitz (London: Academic)
- [20] Sasamoto T M and Wadati M 1999 *J. Phys. Soc. Japan* **66** 2618
- [21] Popkov V, Fouladvand M E and Schütz G M 2002 *J. Phys. A: Math. Gen* **35** 7187
- [22] Popkov V and Schütz G M 2002 *Mat. Fiz. Anal. Geom.* **9** 401 (Preprint hep-th/0302094)
- [23] Yang C N and Yang C P 1966 *Phys. Rev.* **150** 321
- [24] Zamolodchikov A B and Fateev V 1980 *Sov. J. Nucl. Phys.* **32** 298
- [25] Izergin A G and Korepin V E 1981 *Commun. Math. Phys.* **79** 303
- [26] Alcaraz F C and Bariev R Z 2001 *J. Phys. A: Math. Gen.* **34** L467
- [27] Schlottmann P 1987 *Phys. Rev. B* **36** 5177
- [28] Sutherland B 1975 *Phys. Rev. B* **12** 3795
- [29] Perk J H H and Schultz C L 1981 *Phys. Lett. A* **84** 407
- [30] Lieb E H and Wu F Y 1968 *Phys. Rev. Lett.* **20** 1445
- [31] Zamolodchikov A B 1979 *Commun. Math. Phys.* **69** 165
- [32] Zamolodchikov A B and Zamolodchikov A I B 1979 *Ann. Phys., NY* **120** 253
- [33] Alcaraz F C and Bariev R Z 1999 *J. Phys. A: Math. Gen.* **32** L483
- [34] Alcaraz F C and Bariev R Z 1999 *Statistical Physics in the Eve of the 21st Century (Series on Advances in Statistical Mechanics)* vol 14, ed M T Batchelor and L T Wille (Singapore: World Scientific) (Preprint cond-mat/9904042)
- [35] Yang C N 1968 *Phys. Rev. Lett.* **19** 1312
- [36] Kulish P P and Reshetikhin N Yu 1981 *Sov. Phys. -JETP* **53** 108
- [37] Alcaraz F C and Lazo M J to be published
- [38] Essler F H L, Korepin V E and Schoutens K 1992 *Phys. Rev. Lett.* **68** 2960
Essler F H L, Korepin V E and Schoutens K 1993 *Phys. Rev. Lett.* **70** 73
- [39] Alcaraz F C and Bariev R Z 2000 *Braz. J. Phys.* **30** 13